# A geometrical Lagrangian for the neutral scalar meson field



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# A geometrical lagrangian for the neutral scalar meson field

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**Abstract.** A lagrangian derivation of the field equations expressing the gravitational effect of a non-gravitational field usually necessitates the introduction of a field external to the geometry. It is shown, that in the case of a neutral scalar meson field, a geometrical lagrangian exists which yields the Einstein equations for such a field. The result is singular in that the method used here could not be used to derive a geometrical lagrangian for any other non-gravitational field.

## 1. Introduction

Research by Rainich (1925) and independent work by Misner and Wheeler (1957) showed that the Einstein–Maxwell equations can be expressed in completely geometric terms. The resulting equations express both algebraic and differential relations between the components of the Einstein tensor. This procedure can in fact be carried out for any non-gravitational field with a symmetric stress energy tensor and the process has become known as Rainich geometrization. According to Misner and Wheeler, Rainich geometrization effectively geometrizes physics and obviates the need for any sort of unified field theory. One defect of Rainich geometrization is the absence of a lagrangian from which the field equations may be derived. This singular aspect of the field equations means that there is no guarantee of their compatibility. Nevertheless the neutral scalar meson field, whose Rainich geometrization was effected by Kuchar (1963) is an exception. We have been able to derive an expression for a lagrangian which yields field equations conformally equivalent to the field equations expressing the gravitational effect of such a field, ie

$$G_{ij} = -\{\chi_{,i}\chi_{,j} - \tfrac{1}{2}\chi_{,k}\chi^{,k}g_{ij} - \tfrac{1}{2}k^2\chi^2 g_{ij}\}, \tag{1}$$

where

$$\Box\chi = k^2\chi. \tag{2}$$

The signature of space–time is taken to be $(+++-)$, this convention and others (definition of Riemann–Christoffel and Ricci tensors, etc) being the same as that given in Synge (1966). $\Box$ is the d'Alembertian operator in a curved space–time.

## 2. Derivation of the lagrangian

Consider the lagrangian $L = \phi(R)$. Buchdahl (1970) has derived the field equations based on the variation of the action integral

$$I = \int \phi(R)\surd(-g)\,\mathrm{d}^4x.$$

They are:

$$(-R_{,i}R_{,j}+g_{ij}R_{,k}R^{,k})\dddot{\phi}+(-R_{;ij}+g_{ij}\Box R)\ddot{\phi}-R_{ij}\dot{\phi}+\tfrac{1}{2}g_{ij}\phi = 0. \tag{3}$$

The dots signify differentiation with respect to $R$. The substitution $\psi = \ln \epsilon\dot{\phi}$, where $\epsilon = \operatorname{sgn} \dot{\phi}$ casts the equations into the form

$$R_{ij} = -\psi_{;ij}-\psi_{,i}\psi_{,j}+g_{ij}(\Box\psi+\psi_{,k}\psi^{,k})+\tfrac{1}{2}\epsilon g_{ij}\phi\,\mathrm{e}^{-\psi}. \tag{4}$$

When one makes the conformal transformation $g'_{ij} = \mathrm{e}^{\psi}g_{ij} = \epsilon\dot{\phi}g_{ij}$ the signature of space–time is unaltered and the field equations read (cf Synge 1966, p 318)

$$R'_{ij} = -\tfrac{3}{2}\psi_{,i}\psi_{,j}+\tfrac{3}{2}g_{ij}\Box'\psi+\tfrac{1}{2}\epsilon g'_{ij}\phi\,\mathrm{e}^{-2\psi} \tag{5}$$

($R'_{ij}$ and $\Box'$ are the Ricci tensor and d'Alembertian operator in the new space).

In terms of the Einstein tensor these equations are

$$G'_{ij} = -\tfrac{3}{2}(\psi_{,i}\psi_{,j}-\tfrac{1}{2}g'_{ij}g'^{kl}\psi_{,k}\psi_{,l}+g'_{ij}\Box'\psi)-\tfrac{1}{2}\epsilon g'_{ij}\phi\,\mathrm{e}^{-2\psi}. \tag{6}$$

The contraction of equation (5) yields an equation involving $\Box'\psi$, namely,

$$R' = -\tfrac{3}{2}\psi_{,k}\psi_{,l}g'^{kl}+6\Box'\psi+2\epsilon\phi\,\mathrm{e}^{-2\psi}. \tag{7}$$

Both $\phi$ and $\psi$ are defined in terms of $R$, the scalar curvature of the original space. If we express $R'$ in terms of $R$ by means of the equation

$$R' = \mathrm{e}^{-\psi}R+3\Box'\psi-\tfrac{3}{2}g'^{kl}\psi_{,k}\psi_{,l}$$

(cf Synge 1966, p 318) we obtain

$$3\Box'\psi+2\epsilon\phi\,\mathrm{e}^{-2\psi}-R\,\mathrm{e}^{-\psi} = 0. \tag{8}$$

We can now see that if we constrain $\phi$ to satisfy

$$2\epsilon\phi\,\mathrm{e}^{-2\psi}-R\,\mathrm{e}^{-\psi}+3k^2\psi = 0 \tag{9}$$

then $\psi$ must satisfy the Klein–Gordon equation

$$\Box'\psi = k^2\psi. \tag{10}$$

The equation (9) is in fact a first order differential equation for the lagrangian $\phi(R)$. This is more apparent if we substitute $\psi = \ln \epsilon\dot{\phi}$ and multiply by $\epsilon\,\mathrm{e}^{2\psi}$ to obtain

$$3\epsilon k^2\dot{\phi}^2 \ln(\epsilon\dot{\phi})-R\dot{\phi}+2\phi = 0. \tag{11}$$

Because of the term $\ln(\epsilon\dot{\phi})$ it does not seem possible to obtain exact solutions to this equation. However, as a glance at equations (6), the field equations, will show, it is only necessary to obtain an expression for $\phi\,\mathrm{e}^{-2\psi} = \phi/\dot{\phi}^2$ in order to deduce the final form that the field equations take. To this end we differentiate the above equation with respect to $R$, after dividing by $\dot{\phi}$ and obtain the following equation:

$$p\,\mathrm{d}R+[3k^2p(2\ln p+1)-R]\,\mathrm{d}p = 0, \tag{12}$$

where $p = \epsilon\,\dot{\phi} = \mathrm{e}^{\psi}$.

This readily yields the integral

$$\frac{R}{p}+3k^2(\ln p)^2+3k^2\ln p = \text{constant} = A, \tag{13}$$

or, in terms of $\psi$,

$$R\,\mathrm{e}^{-\psi}+3k^2(\psi^2+\psi) = A. \tag{14}$$

From (10), the original differential equation for $\phi$,

$$\epsilon\phi\,\mathrm{e}^{-2\psi} = \tfrac{1}{2}R\,\mathrm{e}^{-\psi}-\tfrac{3}{2}k^2\psi^2$$

and using (14) we finally obtain

$$\epsilon\phi\,\mathrm{e}^{-2\psi} = A-\tfrac{3}{2}k^2\psi^2-3k^2\psi. \tag{15}$$

When we substitute this expression into the field equations (6) at the same time using the equation satisfied by $\psi$ (equation (10)), we obtain

$$G'_{ij} = -\tfrac{3}{2}(\psi_{,i}\psi_{,j}-\tfrac{1}{2}g'_{ij}g'^{kl}\psi_{,k}\psi_{,l}-\tfrac{1}{2}k^2\psi^2g'_{ij})-\tfrac{1}{2}Ag'_{ij}. \tag{16}$$

Making the identification $\chi = \sqrt{\tfrac{3}{2}}\psi$ we can see that equations (16) are identical with (1) the scalar meson field equations save for the cosmological term $-\tfrac{1}{2}Ag'_{ij}$ appearing in (16). $A$ is not in fact a freely specifiable constant. Equation (13), where it first appears, was obtained from (11), the original differential equation for $\phi$, by differentiation and subsequent integration. We shall show, however, that there does exist a solution for $\phi$ in which $A$ equals zero. This means that a cosmological term is not necessary in our equations.

At this point we should also make the following comment. It appears that by varying the lagrangian $\phi(R)$ with respect to the ten $g_{ij}$, we have obtained eleven independent equations, equations (16) above and equation (10) $\Box'\psi = k^2\psi$. This is not in fact the case, for equation (10) may be derived from equations (16) by taking their covariant divergence and thus is not independent. We merely used equation (10) to guide us to (9), the differential equation for the lagrangian.

## 3. Series solution for $\phi$

If we impose the condition $\phi(0) = 0$ (ie we do not admit any cosmological term into the lagrangian) then equation (11) implies that $\epsilon\dot{\phi}(0)$ is either equal to zero or one. If $k = 0$, then $\dot{\phi}(0)$ is in fact the only choice since the relevant solution of (11) is then $\phi = R^2$ and $A = \pm\tfrac{1}{2}$ depending upon the sign of $R$. Field equations based on $R^2$ have been investigated by Buchdahl (1962) and found to be unsatisfactory in that they do not yield asymptotically-flat spherically-symmetric solutions. There is more content to field equations based upon $R^2$ than has hitherto been supposed, however, and this will be the subject of a further paper by the author. When $k$ is not equal to zero, nevertheless, it is possible to show that a solution of (11) exists in which $\epsilon\dot{\phi}(0) = 1$ and that as a consequence of equation (13) $A = 0$.

With the interpretation that the field equations (16) represent the gravitational effect of a meson field then $k$, related to the meson mass by $k = mc/\hbar$, has the dimensions of an inverse length and is of the order of $10^{14}\,\mathrm{cm}^{-1}$ for naturally occurring mesons. Accordingly, in seeking solutions of (11), we shall expand $\phi$ in the series

$$\phi = R + \sum_{n=1}^{\infty} \frac{1}{k^{2n}} f_n(R) \tag{17}$$

valid for $k$ in the neighbourhood of infinity.

$\phi = R$ is, of course, the solution of (11) for $k$ infinite. Since $\epsilon = \operatorname{sgn} \dot{\phi} = 1$ for $k = \infty$ then we shall assume it remains so in the neighbourhood of $\infty$ as we expand $\ln \dot{\phi}$ in the following series:

$$\begin{aligned} \ln \dot{\phi} &= \ln[1-(1-\dot{\phi})] \\ &= -(1-\dot{\phi}) - \tfrac{1}{2}(1-\dot{\phi})^2 - \tfrac{1}{3}(1-\dot{\phi})^3 - \ldots \\ &= \frac{1}{k^2}\dot{f}_1 + \frac{1}{k^4}(\dot{f}_2 - \tfrac{1}{2}\dot{f}_1^2) + \ldots . \end{aligned} \tag{18}$$

Substitution of equation (18) into (11) then yields the series for $\phi$

$$\phi = R - \frac{R^2}{6k^2} - \frac{R^3}{18k^4} - \ldots .$$

Thus we have accomplished our stated aim: the derivation of a geometrical lagrangian from which the Einstein equations for a neutral scalar meson field may be derived.

## 4. Conclusions

Qualitatively, the above lagrangian represents the standard Einstein lagrangian $R$ together with correction terms which become more important as the scalar curvature increases, ie as we approach the source of the field. It is satisfying to have such a lagrangian since it means that quantities external to the geometry do not have to be introduced in order to derive the field equations. On the other hand, we acknowledge that it is rather difficult to see any *a priori* justification for the lagrangian derived here. A point worth noting is that it would not be possible to derive a lagrangian for, say, the Einstein–Maxwell equations on their Rainich geometrized equivalent by the method used in this paper. Our lagrangian is based upon an inherently geometrical scalar field, namely, the scalar curvature $R$. There is no such naturally occurring vector field in riemannian geometry which would define a similar lagrangian for the Einstein–Maxwell equations.

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